# On the Peaking Attenuation and Transient Response Improvement of High-Gain Observers

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***Abstract*— In this paper a novel method is proposed for state estimation of nonlinear systems using high-gain observers (HGOs) and adaptive techniques. In this regard, Multiple HGOs (MHGO) are run simultaneously, and the information obtained from individual observers are combined adaptively. To be able to suitably combine the state estimations, it is first proved that there exist some constant coefficients that provide the perfect estimation. Then, the RLS algorithm is employed to find those coefficients. The convergence of the state estimations to the system states is guaranteed, and it is shown that the MHGO is able to attenuate the inherent peaking phenomenon in HGOs. Finally, the simulation results are presented which show the superiority of the proposed MHGO method in state estimation and improving the transient response.**

## I. Introduction

High-Gain Observer (HGO) is a powerful tool used in control theory [1], [2]. In the past few decades, a wide range of control problems are solved using this structure, e.g., fault detection and isolation [3], control of nonlinear systems [4], parameter estimation [5]. One of the unique features of HGO that has brought such a broad attention to it is that the separation principal holds in HGO-based control strategies. This condition is first discussed in details in [6], [7], and some improvements/extensions are made on that in [8]. In spite of this fact, an inherent drawback of HGO is the existence of undesired peaks in the transient response of the estimated states, known as the peaking phenomenon. In [9], this issue is studied and it is shown that the interaction of this behaviour with system nonlinearities could induce destructive effects on the performance of the overall system.

It is well-known that transient response of adaptive systems is oscillatory during the learning phase, which is not preferable from practical point of view [10]. To overcome this problem, many investigations have been launched. The idea of utilizing multiple models is used in [11] to identify the plant dynamics rapidly. In this scheme a performance criteria is defined based on which one of the identified models is selected at any instant. However, to access an accurate/satisfactory identification, it is required to use a large number of models. Moreover, another drawback is that the employed models do not cooperate with each other. For linear time invariant systems, a novel scheme, called second-level adaptation method, is presented in [12] that utilizes the information obtained from multiple parameter estimators to overcome the above challenges. This approach is able to improve the transient response since it uses a convex combination of the individual estimators' information (information is fused at any instant). In the past several years, the second-level adaptation method attracted the attention, and many researchers started utilizing this concept in solving the problems raised in system theory. In [13], this idea is employed to design a fast and smooth adaptive controller for LTI systems having parameter uncertainties. Control problem of a class of nonlinear systems with large parametric uncertainties is studied in [14], in which multiple models with second-level adaptation are used for identification purposes that resulted in a better transient performance as well as fast convergence.

In this paper, HGO's estimation capabilities are combined together with the second-level adaptation to reconstruct the states of nonlinear systems rapidly and accurately. The main contributions of this paper are: I) A new observation methodology is proposed for nonlinear systems using multiple HGOs (MHGO) and second-level adaptation method. II) Unlike the previously developed second-level adaptation-based methods, which are aimed at estimating unknown *constant parameters*, the proposed structure deals with the state estimation problem. III) The existence of some *unknown constant* parameters enabling us to estimate the system states exactly is guaranteed. IV) The stability of the proposed method as well as the convergence of state estimations to system states are proved. V) It is proved that MHGO can proved an estimation with more preferable transient response in comparison to single HGO.

## II. The Main Results

This section consists of problem formulation and its solution using HGO. Then, the structure of the proposed MHGO is introduced, and the existence of ideal state estimation using the proposed strategy is discussed. Afterwards, the stability of MHGO and the convergence of state estimations to plant states is shown. Then, the performance of the proposed method is investigated.

### A. State Estimation of Nonlinear Systems by HGO

Consider the following nonafine SISO nonlinear system

$$\begin{aligned} \dot{x} &= Ax + Bf(x,u) \\ y &= Cx \end{aligned} \tag{1}$$

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where $x \in R^n$ is the state vector, $u \in R$ is the input signal, $y \in R$ is the output of system, and $f(x,u)$ is a nonlinear function. The $n \times n$ matrix $A$, the $n \times 1$ vector $B$, and the $1 \times n$ vector $C$ are given by

$$A = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}, B = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix}$$

It is assumed that $f(x,u)$ is locally Lipschitz in $(x,u)$ for all $x \in \mathcal{D} \subseteq \mathcal{R}^n$ and $u \in \mathcal{B} \subseteq \mathcal{R}$, and $f(0,0) = 0$. Thus, the origin is an equilibrium point of the system [8].

In order to present the proposed method, we need to provide the principle of HGO. A HGO that is able to estimate the states of (1), $\hat{x}$, has the following form [2], [15]:

$$\dot{\hat{x}} = A\hat{x} + Bf_0(\hat{x},u) + H(y - C\hat{x}) \tag{2}$$

with $H = [\kappa_1/\epsilon \quad \kappa_2/\epsilon^2 \quad \cdots \quad \kappa_n/\epsilon^n]^T$, where $\epsilon \in (0,1]$ and $\kappa_i$s are chosen such that the roots of $s^n + \kappa_1 s^{n-1} + \cdots + \kappa_{n-1}s + \kappa_n = 0$ are in the open left-half plane. This guarantees that $A - HC$ is a Hurwitz matrix. The nonlinear function $f_0(\cdot)$ is a saturated version of $f(\cdot)$ which agrees with $f(\cdot)$ on the domain of interest, i.e., $\mathcal{D} \times \mathcal{B}$ [15].

Now by defining estimation error as $\tilde{x} = x - \hat{x}$, its dynamics can be obtained by subtracting (2) from (1) as follows:

$$\dot{\tilde{x}} = A_0\tilde{x} + B(f(x,u) - f_0(\hat{x},u)) \tag{3}$$

where $A_0 = A - HC$. It can be shown that there exists $\epsilon_0^* > 0$ such that, for every $0 < \epsilon \leq \epsilon_0^*$ and any admissible $x \in \mathcal{D}$ and $u \in \mathcal{B}$, the effect of $f(x,u) - f_0(\hat{x},u)$ on $\tilde{x}$ is rejected. According to the fact that $A_0$ is a Hurwitz matrix, it can be shown that $\lim_{t\to\infty} \tilde{x}(t) = 0$ [2], [15].

### B. The Structure of the Proposed MHGO

Before presenting the proposed structure, it is required to consider the following Lemma.

*Lemma 1 ([16]):* Let $\mathcal{Q}$ be a set in a linear space $\mathcal{L}$. The convex hull $\mathcal{K}$ of $\mathcal{Q}$ is the smallest convex set containing $\mathcal{Q}$. For $a_i \in \mathcal{Q}$ with $i = 1, \cdots, m$, the convex hull of $a_i$s is given by $\mathcal{K} = \{\sum_{i=1}^{m} \beta_i a_i\}$ for $\beta_i \geq 0$ and $\sum_{i=1}^{m} \beta_i = 1$.

The goal is to obtain an state estimation that uses information provided by multiple HGOs in the sense of convex combination. Toward this end, we propose the employing of $N$ high-gain observers with the following structure

$$\begin{aligned} &\dot{\hat{x}}_i(\alpha,t) = A\hat{x}_i(\alpha,t) + H(y(t) - C\hat{x}_i(\alpha,t)) \\ &+Bf_0(\sum_{i=1}^{N} \alpha_i\hat{x}_i(\alpha,t), u(t)), \quad \hat{x}_i(\alpha,0) = \hat{x}_i(0) \end{aligned} \tag{4}$$

where $i = 1, \cdots, N$, $\hat{x}_i$ is state estimation from the $i$th observer, $\alpha_i \geq 0$ such that $\sum_{i=1}^{N} \alpha_i = 1$, and $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{bmatrix}^T$. The notation $\hat{x}_i(\alpha)$ is chosen to show that $\hat{x}_i$s are dependent on $\alpha$. In order to employ the obtained $N$ state estimations, the following combination of them is considered as the final state estimation

$$\hat{x}_o(t) = \sum_{i=1}^{N} \alpha_i\hat{x}_i(\alpha,t) \tag{5}$$

As it can be seen, the proposed method utilizes the obtained estimations from $N$ observers for providing a more accurate state estimation. On the other hand, it is required to prove that this structure is able to yield preferable estimations, which is the goal of subsequent sections.

### C. Existence of Ideal State Estimation

This section investigates the capability of the proposed observer structure (4) in estimating actual system states. Toward this end, we need to show that there exist constant $\alpha_i^*$s so that when $\alpha_i$s are set equal to them, the ensuing estimation satisfies the equality of $x(t) = \hat{x}_o(t)$ for all $t \geq 0$. To achieve this goal, the analysis is divided into two stages. First, it is needed to show that there exist $\alpha_i^*$s such that $x(t = 0) = \hat{x}_o(t = 0)$ if $\alpha_i = \alpha_i^*$s. Next, it will be also shown that the equality, $x(t) = \hat{x}_o(t)$, holds for all $t > 0$.

**First Step ($t = 0$):** In order to accomplish the first step of the proof, it is required to consider the properties of convex hull of a set in linear space. Then, by employing Lemma 1, it can be seen that if $\hat{x}_i(0)$s are chosen so that $x(0)$ is in the convex hull $\mathcal{K}$ of $\hat{x}_i(0)$s, then some $\alpha_i^*$s exist such that

$$x(0) = \sum\nolimits_{i=1}^{N} \alpha_i^*\hat{x}_i(0) \tag{6}$$

This concludes the first step. Also, note that for the satisfaction of the preceding equality, at least $N = n + 1$ observers are required; where $n$ is the number of state variables.

**Second Step ($t > 0$):** Since this step is aimed at providing the analysis for $t > 0$, it is required to obtain the observation error dynamics. Subtracting (4) from (1), results in

$$\begin{aligned} \dot{\tilde{x}}_i(\alpha,t) &= A_0\tilde{x}_i(\alpha,t) \\ &+ B(f(x(t),u(t)) - f_0(\sum_{i=1}^{N} \alpha_i\hat{x}_i(\alpha,t), u(t)) \end{aligned} \tag{7}$$

where $\tilde{x}_i = x - \hat{x}_i$ denotes observation error of the $i$th observer. To obtain the error dynamics of MHGO, one can use (5), $\sum_{i=1}^{N} \alpha_i = 1$, and $\tilde{x}_o = x - \hat{x}_o$ to get $\tilde{x}_o(t) = \sum_{i=1}^{N} \alpha_i\tilde{x}_i(\alpha,t)$. Since $\alpha_i$s are constants, taking the time derivative of the preceding equation and using (7) yields

$$\dot{\tilde{x}}_o = A_0\tilde{x}_o + B(f(x,u) - f_0(\hat{x}_o,u)) \tag{8}$$

The next lemma is presented to wrap the second step up.

*Lemma 2:* Let for (1), $x \in \mathcal{D}$, $u \in \mathcal{B}$, and the initial conditions of HGOs (4) be chosen such that $x(0)$ is in their convex hull $\mathcal{K}$. Then there exist some $\alpha_i^*$s such that by choosing $\alpha_i = \alpha_i^*$s, the obtained state estimation from (5) is equal to $x(t)$ for all $t \geq 0$.

**Proof.** Consider scaled estimation error

$$\eta_{(i)} = \frac{x_{(i)} - \hat{x}_{o(i)}}{\epsilon^{n-i}}, i = 1, \cdots, n$$

where $x_{(i)}$ and $\hat{x}_{o(i)}$ are the $i$th elements of $x$ and $\hat{x}_o$, respectively.Therefore, we have

$$\begin{aligned} &D(\epsilon)\eta = x - \hat{x}_o \\ &D(\epsilon) = \text{diag}(\epsilon^{n-1}, \epsilon^{n-2}, \cdots, \epsilon, 1) \end{aligned} \tag{9}$$

where $\eta = [\eta_{(1)} \quad \cdots \quad \eta_{(n)}]^T$. Using (8) and (9), one has

$$\epsilon\dot{\eta} = A_1\eta + \epsilon B(f(x,u) - f_0(\hat{x}_o,u)) \tag{10}$$

where $A_1 = \epsilon D(\epsilon)^{-1}A_0D(\epsilon)$. Note that since $\kappa_i$s are chosen such that $A_0$ is a Hurwitz matrix, $A_1$ is also a Hurwitz matrix. With regard to this fact, one can consider a Lyapunov function candidate $V(\eta) = \eta^T P_1 \eta$ where $P_1$ is a positive definite matrix satisfying $A_1^T P_1 + P_1 A_1 = -I$. By taking the time derivative of $V(\eta)$ and using (10), we have

$$\dot{V}(\eta) = -\frac{1}{\epsilon}\eta^T\eta + 2B^T(f(x,u) - f_0(\hat{x}_o,u))P_1\eta \tag{11}$$

Since in the domain $\mathcal{D}\times\mathcal{B}$, $f_0$ agrees with $f$, and both are locally Lipschitz, one can get

$$\|f(x,u) - f_0(\hat{x}_o,u)\| \le L_1\|\eta\| \tag{12}$$

where $L_1 > 0$ is a Lipschitz constant. It is worth noting that $\|D(\epsilon)\| = 1$ is used for obtaining the previous equation. By using (12) and $\|B\| = 1$, $\dot{V}(\eta)$ can be expressed as

$$\dot{V}(\eta) \le -\frac{1}{\epsilon}\|\eta\|^2 + 2L_1\|P_1\|\|\eta\|^2 \tag{13}$$

The following equation can be considered

$$\lambda_{\min}(P_1)\|\eta\|^2 \le V(\eta) \le \lambda_{\max}(P_1)\|\eta\|^2 \tag{14}$$

where $\lambda_{\min}(P_1)$ and $\lambda_{\max}(P_1)$ are the smallest and the largest eigenvalues of $P_1$, respectively. Now, by using (13) and (14), it can be shown that $\dot{V}(\eta) \le k_1 V(\eta)$ where $k_1 = -\frac{1}{\epsilon\lambda_{\max}(P_1)} + 2\frac{L_1}{\lambda_{\min}(P_1)}\|P_1\|$. Therefore, one has

$$V(t) \le e^{k_1 t}V(0). \tag{15}$$

Furthermore, one can get the following inequality by incorporating (14) and (15).

$$\lambda_{\min}(P_1)\|\eta(t)\|^2 \le e^{k_1 t}\lambda_{\max}(P_1)\|\eta(0)\|^2 \tag{16}$$

On the other hand, from (9) it is obtained

$$\epsilon^{2(n-1)}\|\eta\|^2 \le \|x - \hat{x}_o\|^2 \le \|\eta\|^2. \tag{17}$$

Finally, using (16) and (17) and performing some basic manipulations result in

$$\|x(t) - \hat{x}_o(t)\| \le \frac{k_2}{\epsilon^{n-1}}e^{\frac{1}{2}k_1 t}\|x(0) - \hat{x}_o(0)\| \tag{18}$$

where $k_2 = \sqrt{\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}$. Since $\hat{x}_i(0)$s are chosen such that $x(0)$ is in their convex hull, there exist $\alpha_i = \alpha_i^*$s such that (6) is valid (refer to Lemma 1). Consequently, the right hand side of (18) is zero, which completes the proof. □

From Lemma 2 it can be realized that if $x(0)$ is in the convex hull $\mathcal{K}$ of $\hat{x}_i(0)$s, there are some positive constant $\alpha_i = \alpha_i^*$s that provide perfect state estimation. This property can be utilized to transform the problem of state estimation to estimation of some constant parameters.

### D. Finding $\alpha_i$s

Till now, the conditions based on which there exist $\alpha_i^*$s such that the perfect state estimation can be achieved are provided. Now, by choosing $\alpha_i = \alpha_i^*$s, we have

$$x(t) = \sum_{i=1}^{N}\alpha_i^*\hat{x}_i(\alpha^*,t), \forall t \ge 0 \tag{19}$$

where $\alpha^* = \begin{bmatrix}\alpha_1^* & \alpha_2^* & \cdots & \alpha_N^*\end{bmatrix}^T$. However, one cannot use this selection since $\alpha_i^*$s are unknown. Hence obtaining an appropriate estimation of $\alpha_i^*$s is needed for state estimation. In this regard, let us use $\sum_{i=1}^N \alpha_i^* = 1$ and rearrange (19) as $\sum_{i=1}^N \alpha_i^*\tilde{x}_i(\alpha^*,t) = 0$. Furthermore, one can consider $\alpha_N^* = 1 - \sum_{i=1}^{N-1}\alpha_i^*$ and get

$$\sum_{i=1}^{N-1}\alpha_i^*(\tilde{x}_i(\alpha^*,t) - \tilde{x}_N(\alpha^*,t)) = -\tilde{x}_N(\alpha^*,t) \tag{20}$$

By using $\tilde{x}_i(\alpha^*,t) - \tilde{x}_N(\alpha^*,t) = \hat{x}_N(\alpha^*,t) - \hat{x}_i(\alpha^*,t)$ and (4), it can be obtained

$$\dot{\hat{x}}_N(\alpha^*,t) - \dot{\hat{x}}_i(\alpha^*,t) = A_0(\hat{x}_N(\alpha^*,t) - \hat{x}_i(\alpha^*,t)) \tag{21}$$

with $A_0 = A - HC$. From the preceding equation one can see that $\hat{x}_N(\alpha^*,t) - \hat{x}_i(\alpha^*,t)$ is the state of a linear system with the initial condition of $\hat{x}_N(0) - \hat{x}_i(0)$. Therefore, it is independent of $\alpha^*$, and if a matrix $E$ is defined such that its $i$th column is $\hat{x}_N(\alpha^*,t) - \hat{x}_i(\alpha^*,t)$ , it is also independent of $\alpha^*$. By utilizing this fact and defining $\bar{\alpha}^* = [\alpha_1^* \quad \cdots \quad \alpha_{N-1}^*]^T$, one can rewrite (20) as follows

$$E(t)\bar{\alpha}^* = -\tilde{x}_N(\bar{\alpha}^*,t) \tag{22}$$

Since $x$ and $\hat{x}_N(\bar{\alpha}^*,t)$ are unknown, one cannot employ the previous equation for estimating $\bar{\alpha}^*$. For solving this problem, we premultiply (22) by $C$ to get

$$CE(t)\bar{\alpha}^* = -\tilde{y}_N(\bar{\alpha}^*,t) \tag{23}$$

where $\tilde{y}_N(\bar{\alpha}^*,t) = y(t) - C\hat{x}_N(\bar{\alpha}^*,t)$. In the previous equation, $\hat{x}_N(\bar{\alpha}^*,t)$ is still unknown, which makes the utilization of the Recursive Least Squares (RLS) algorithm for estimating $\bar{\alpha}^*$ impossible since it needs $\tilde{y}_N(\bar{\alpha}^*,t)$. However, we propose the utilization of a modified RLS algorithm as follows

$$\begin{aligned}\dot{\hat{\bar{\alpha}}} &= -PE^TC^T(\tilde{y}_N(\hat{\bar{\alpha}}) + CE\hat{\bar{\alpha}}), & \hat{\bar{\alpha}}(0) &= \hat{\bar{\alpha}}_0\\ \dot{P} &= -PE^TC^TCEP, & P(0) &= \gamma I\end{aligned} \tag{24}$$

where $\hat{\bar{\alpha}}$ is the estimation of $\bar{\alpha}^*$, $\tilde{y}_N(\hat{\bar{\alpha}}) = y - C\hat{x}_N(\hat{\bar{\alpha}})$, $I$ is the identity matrix, and $\gamma$ is a positive constant. In addition, using the fact that $E(t)$ is independent of $\bar{\alpha}^*$, its $i$th column can be obtained using $\hat{x}_N(\hat{\bar{\alpha}},t) - \hat{x}_i(\hat{\bar{\alpha}},t)$. It will be shown that the modified RLS algorithm is able to provide an estimation of $\bar{\alpha}^*$ which is appropriate for estimating $x(t)$.

Finally, the state estimation is calculated as follows

$$\begin{aligned}\dot{\hat{x}}_i(\hat{\bar{\alpha}}) &= A\hat{x}_i(\hat{\bar{\alpha}}) + H(y - C\hat{x}_i(\hat{\bar{\alpha}})) + Bf_0(\hat{x}_o,u)\\ \hat{x}_o &= \sum_{i=1}^{N-1}\hat{\alpha}_i\hat{x}_i(\hat{\bar{\alpha}}) + (1 - \sum_{i=1}^{N-1}\hat{\alpha}_i)\hat{x}_N(\hat{\bar{\alpha}})\end{aligned} \tag{25}$$

By considering (24) and (25), it is obvious that the proposed MHGO is consisted of two interconnected systems. Consequently, it is required to investigate its stability and estimation converges, which is the goal of following theorem.

*Theorem 1:* Let for (1), $x \in \mathcal{D}$ and $u \in \mathcal{B}$. Then for (24) and (25), there exist some $\epsilon^* > 0$ such that by choosing $0 < \epsilon \leq \epsilon^*$, the obtained state estimation from (25) converges to $x$. In addition, $\hat{x}_i$s are uniformly ultimately bounded, and $\hat{\bar{\alpha}}$ and $P$ are bounded.

**Proof.** Consider the scaled error (9), hence one can define $D\eta_i = x - \hat{x}_i$ and use (25) and $\sum_{i=1}^{N} \hat{\alpha}_i = 1$ to obtain

$$\begin{aligned} \epsilon\dot{\eta}_i(\hat{\bar{\alpha}}) &= A_1\eta_i(\hat{\bar{\alpha}}) + \epsilon B[f(x,u) - f_0(x - D(\epsilon)\eta, u)] \\ \eta &= \sum_{i=1}^{N-1} \hat{\alpha}_i\eta_i(\hat{\bar{\alpha}}) + (1 - \sum_{i=1}^{N-1} \hat{\alpha}_i)\eta_N(\hat{\bar{\alpha}}) \end{aligned} \tag{26}$$

In addition, by defining $DE = E_1$ and using $CD(\epsilon) = \epsilon^{n-1}C$, the RLS algorithm (24) is considered as

$$\begin{aligned} \dot{\hat{\bar{\alpha}}} &= -\epsilon^{2(n-1)} PE_1^T C^T C(\eta_N(\hat{\bar{\alpha}}) + E_1\hat{\bar{\alpha}}) \\ \dot{P} &= -\epsilon^{2(n-1)} PE_1^T C^T CE_1 P \end{aligned} \tag{27}$$

It is worth noting that the $i$th column of $E_1$ is equal to $\eta_i(\hat{\bar{\alpha}}) - \eta_N(\hat{\bar{\alpha}})$. Therefore, it is valid to say that

$$\epsilon\dot{E}_1 = A_1E_1 \tag{28}$$

Using the fact that $A_1$ is Hurwitz, it can be shown that there exist positive constants $k_1$ and $\lambda_1$ such that

$$\|e^{\frac{1}{\epsilon}A_1t}\| \leq k_1e^{-\frac{\lambda_1}{\epsilon}t} \tag{29}$$

One can conclude from the preceding equation and (28) that

$$\|E_1(t)\| \leq k_1\|E_1(0)\|e^{-\frac{\lambda_1}{\epsilon}t} \tag{30}$$

In order to prove the boundedness of $P(t)$, one can consider the facts that it is a positive definite matrix and $\dot{P}(t) \leq 0$. Hence, $P(t) \leq P(0)$, i.e., $P(t)$ is bounded.

Now, for showing the stability of MHGO, let us to employ (26) and the definition of $E_1$ to obtain

$$\eta = E_1\hat{\bar{\alpha}} + \eta_N(\hat{\bar{\alpha}}) \tag{31}$$

Therefore, one can employ (27) and get

$$\dot{\hat{\bar{\alpha}}} = -\epsilon^{2(n-1)} PE_1^T C^T C\eta \tag{32}$$

For obtaining the observation error system, one needs to obtain $\dot{\eta}$ from (31) and use (26), (28), and (32), which follows

$$\begin{aligned} \dot{\eta} = \frac{1}{\epsilon}A_1\eta &- \epsilon^{2(n-1)}E_1PE_1^TC^TC\eta \\ &+ B[f(x,u) - f_0(x - D(\epsilon)\eta, u)] \end{aligned}$$

One can use the preceding equation and a Lyapunov function candidate $V(\eta) = \eta^T P_1\eta$ with $A_1^TP_1 + P_1A_1 = -I$ to obtain

$$\begin{aligned} \dot{V}(\eta) = &-\frac{1}{\epsilon}\eta^T\eta - 2\epsilon^{2(n-1)}\eta^TP_1E_1PE_1^TC^TC\eta \\ &+ 2\eta^TP_1B[f(x,u) - f_0(x - D(\epsilon)\eta, u)] \end{aligned} \tag{33}$$

If we employ (12), $\|B\| = 1$, and define $\epsilon^* = \frac{1}{4L_1\|P_1\|}$, then by choosing $0 < \epsilon \leq \epsilon^*$ we have

$$\dot{V}(\eta) \leq -\frac{1}{2\epsilon}\|\eta\|^2 + 2\epsilon^{2(n-1)}\|P_1\|\|E_1\|^2\|P\|\|\eta\|^2$$

On the other hand, since $P(t)$ is bounded, it is valid to say that there exists a positive constant $\bar{p}$ such that $\|P(t)\| \leq \bar{p}$. Consequently, using (30) and (14), one has

$$\dot{V}(\eta) \leq (-\frac{1}{2\epsilon\lambda_{\max}(P_1)} + \frac{1}{\lambda_{\min}(P_1)}\rho e^{-2\frac{\lambda_1}{\epsilon}t})V(\eta)$$

where $\rho = 2\epsilon^{2(n-1)}\bar{p}k_1^2\|P_1\|\|E_1(0)\|^2$. From the preceding equation, one can get $V(t) \leq k_2(t)e^{-\frac{1}{2\epsilon\lambda_{\max}(P_1)}t}V(0)$ where $k_2(t) = e^{\frac{\epsilon}{2\lambda_{\min}(P_1)\lambda_1}\rho(1-e^{-2\frac{\lambda_1}{\epsilon}t})}$. It can be seen that $k_2(t) \leq \bar{k}_2$ with $\bar{k}_2 = e^{\frac{\epsilon}{2\lambda_{\min}(P_1)\lambda_1}\rho}$. As a result, we have

$$V(t) \leq \bar{k}_2e^{-\frac{1}{2\epsilon\lambda_{\max}(P_1)}t}V(0) \tag{34}$$

Finally, by using (14), one can get

$$\|\eta(t)\| \leq k_3e^{-\frac{1}{4\epsilon\lambda_{\max}(P_1)}t}\|\eta(0)\| \tag{35}$$

where $k_3 = \sqrt{\bar{k}_2\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}$. Therefore, we have $\lim_{t\to\infty}\eta(t) = 0$; in other words, the obtained state estimation $\hat{x}_o(t)$ converges to the state $x(t)$.

For $\hat{\bar{\alpha}}$, one can take the integral of (32) and obtain

$$\hat{\bar{\alpha}}(t) = \hat{\bar{\alpha}}_0 - \epsilon^{2(n-1)}\int_0^t P(\tau)E_1^T(\tau)C^TC\eta(\tau)d\tau$$

Now, let us to use $\|P\| \leq \bar{p}$, (30), and (35) to get

$$\|\hat{\bar{\alpha}}(t)\| \leq \|\hat{\bar{\alpha}}_0\| + k_4\int_0^t e^{-\frac{1}{\epsilon}(\lambda_1 + \frac{1}{4\lambda_{\max}(P_1)})\tau}d\tau$$

with $k_4 = \epsilon^{2(n-1)}\bar{p}k_1k_3\|E_1(0)\|\|\eta(0)\|$; thus $\hat{\bar{\alpha}}$ is bounded.

For $\hat{x}_i$s, we consider a Lyapunov function candidate $V_i(\eta_i) = \eta_i^TP_1\eta_i$ and employ (26) to obtain

$$\dot{V}_i(\eta_i) \leq -\frac{1}{\epsilon}\|\eta_i\|^2 + 2L_1\|P_1\|\|\eta_i\|\|\eta\|$$

On the other hand, from (35) we have $\|\eta(t)\| \leq k_3\|\eta(0)\|$. Therefore, $\dot{V}_i(\eta_i) < 0$ for $\|\eta_i\| > 2\epsilon L_1k_3\|P_1\|\|\eta(0)\|$, and one can use the boundedness of $x$ and conclude that $\hat{x}_i$s are uniformly ultimately bounded. $\square$

The presented theorem states that MHGO is stable and its estimation converges to the state of the plant. Now, it is required to show that MHGO is also able to provide better estimation in comparison to single HGO. In this regard, one can add $\pm E_1(t)\bar{\alpha}^*$ to the right hand side of (31) and employ (22) to obtain the following equation

$$\eta = E_1\tilde{\bar{\alpha}} + \sigma(\hat{\bar{\alpha}}) \tag{36}$$

where $\tilde{\bar{\alpha}} = \hat{\bar{\alpha}} - \bar{\alpha}^*$ and $\sigma(\hat{\bar{\alpha}}) = \eta_N(\hat{\bar{\alpha}}) - \eta_N(\bar{\alpha}^*)$. In order to analyze the preceding equation, one needs to obtain $\sigma(\hat{\bar{\alpha}})$. Toward this end, from its definition, we have $\sigma(\hat{\bar{\alpha}}(0), 0) = 0$. Moreover, by employing (26) and the fact that $\epsilon\dot{\eta}_i(\bar{\alpha}^*) = A_1\eta_i(\bar{\alpha}^*)$, one can get $\dot{\sigma}(\hat{\bar{\alpha}}) = \frac{1}{\epsilon}A_1\sigma(\hat{\bar{\alpha}}) + B\left[f(x,u) - f_0(x - D\eta, u)\right]$. As a result, it is valid to say

$$\sigma(\hat{\bar{\alpha}}(t),t) = \int_0^t e^{\frac{1}{\epsilon}A_1(t-\tau)}B[f(x(\tau),u(\tau)) - f_0(x(\tau) - D\eta(\tau), u(\tau))]d\tau$$

and by using (12) and (29), we have

$$\|\sigma(\hat{\bar{\alpha}}(t),t)\| \leq \frac{\epsilon}{\lambda_1}k_1L_1 \sup_{0\leq\tau\leq t}\|\eta(\tau)\| \tag{37}$$

Now, we need to obtain $\tilde{\bar{\alpha}}$ in (36); therefore, (32) and (36) are considered to get

$$\dot{\tilde{\bar{\alpha}}} = -\epsilon^{2(n-1)}PE_1^TC^TC(E_1\tilde{\bar{\alpha}} + \sigma(\hat{\bar{\alpha}})) \tag{38}$$

On the other hand, one can consider $\frac{dP^{-1}}{dt} = -P^{-1}\dot{P}P^{-1}$, (27), and (38) to obtain $\frac{d(P^{-1}\tilde{\bar{\alpha}})}{dt} = -\epsilon^{2(n-1)}E_1^TC^TC\sigma(\hat{\bar{\alpha}})$. Then, one can take the integral of previous equation and premultiply it by $P(t)$ to get

$$\begin{aligned}\tilde{\bar{\alpha}}(t) &= P(t)P(0)^{-1}\tilde{\bar{\alpha}}_0 \\ &- \epsilon^{2(n-1)}P(t)\int_0^t E_1(\tau)^TC^TC\sigma(\hat{\bar{\alpha}}(\tau),\tau)d\tau\end{aligned} \tag{39}$$

To obtain $P(t)$ in (39), $\frac{dP^{-1}}{dt} = -P^{-1}\dot{P}P^{-1}$ and (27) can be used to have $\frac{dP^{-1}}{dt} = \epsilon^{2(n-1)}E_1^TC^TCE_1$. By taking the integral of preceding equation, one has $P(t)^{-1} - P(0)^{-1} = \epsilon^{2(n-1)}\int_0^t E_1(\tau)^TC^TCE_1(\tau)d\tau$. Now, one can use $E_1(t) = e^{\frac{1}{\epsilon}A_1t}E_1(0)$ and $P(0) = \gamma I$ to get

$$P(t) = \left[\frac{1}{\gamma}I + \epsilon^{2(n-1)}E_1(0)^TW_o(t)E_1(0)\right]^{-1}$$

where $W_o(t) = \int_0^t e^{\frac{1}{\epsilon}A_1^T\tau}C^TCe^{\frac{1}{\epsilon}A_1\tau}d\tau$ is the observability Gramian. Since the pair $(A_1, C)$ is observable, $W_o(t)$ is a positive definite matrix. On the other hand, to proceed with the analysis, it is assumed that $E_1(0)$ has full column rank. Therefore, $E_1(0)^TW_o(t)E_1(0)$ has full rank; and in turn, by choosing $\gamma$ big enough such that $\gamma\epsilon^{2(n-1)}\lambda_{\min}(E_1(0)^TW_o(t)E_1(0)) \gg 1$ for all $t > 0$, it can be obtained

$$P(t) \approx \frac{1}{\epsilon^{2(n-1)}}\left[E_1(0)^TW_o(t)E_1(0)\right]^{-1}$$

It is worth noting that the preceding equation can be easily shown by employing the Jordan form of $E_1(0)^TW_o(t)E_1(0)$.

Now, substituting $P(t)$ in (39) results

$$\begin{aligned}\tilde{\bar{\alpha}}(t) &\approx \frac{1}{\gamma\epsilon^{2(n-1)}}\left[E_1(0)^TW_o(t)E_1(0)\right]^{-1}\tilde{\bar{\alpha}}_0 \\ &- \left[E_1(0)^TW_o(t)E_1(0)\right]^{-1}\int_0^t E_1(\tau)^TC^TC\sigma(\hat{\bar{\alpha}}(\tau),\tau)d\tau\end{aligned}$$

Since $E_1(0)^TW_o(t)E_1(0)$ has full rank, its inverse is bounded; therefore, there exists a positive constant $b$ such that $\|\left[E_1(0)^TW_o(t)E_1(0)\right]^{-1}\| \leq b$. Now (30) and $\int_0^t e^{-\frac{\lambda_1}{\epsilon}\tau}d\tau \leq \epsilon/\lambda_1$ can be used to obtain

$$\begin{aligned}\|E_1(t)\tilde{\bar{\alpha}}(t)\| &\leq \frac{1}{\gamma\epsilon^{2(n-1)}}k_1b\|E_1(0)\|\|\tilde{\bar{\alpha}}_0\| \\ &+ \frac{\epsilon}{\lambda_1}k_1^2b\|E_1(0)\|^2\sup_{0\leq\tau\leq t}\|\sigma(\hat{\bar{\alpha}}(\tau),\tau)\|\end{aligned} \tag{40}$$

By using (37) and the fact that the supremum of a function is its least upper bound, one can get

$$\sup_{0\leq\tau\leq t}\|\sigma(\hat{\bar{\alpha}}(\tau),\tau)\| \leq \frac{\epsilon}{\lambda_1}k_1L_1\sup_{0\leq\tau\leq t}\|\eta(\tau)\|$$

Now one can employ the preceding equation, (36), and (40) to obtain

$$\begin{aligned}\|\eta(t)\| &\leq \frac{1}{\gamma\epsilon^{2(n-1)}}k_1b\|E_1(0)\|\|\tilde{\bar{\alpha}}_0\| \\ &+ (\frac{\epsilon}{\lambda_1}k_1^2b\|E_1(0)\|^2 + 1)\frac{\epsilon}{\lambda_1}k_1L_1\sup_{0\leq\tau\leq t}\|\eta(\tau)\|\end{aligned}$$

From the previous equation it can be concluded that $\sup\|\eta(\tau)\|$ is also less than the right hand side. On the other hand, there exists $\epsilon_1^* > 0$ such that by chossing $0 < \epsilon < \epsilon_1^*$, the inequality $1 - (\frac{\epsilon}{\lambda_1}k_1^2b\|E_1(0)\|^2 + 1)\frac{\epsilon}{\lambda_1}k_1L_1 > 0$ is valid. Therefore, we obtain

$$\sup_{0\leq\tau\leq t}\|\eta(\tau)\| \leq \frac{\frac{1}{\gamma\epsilon^{2(n-1)}}k_1b\|E_1(0)\|\|\tilde{\bar{\alpha}}_0\|}{1 - (\frac{\epsilon}{\lambda_1}k_1^2b\|E_1(0)\|^2 + 1)\frac{\epsilon}{\lambda_1}k_1L_1} \tag{41}$$

To show that the proposed MHGO provides a better estimation, we need to analyze the estimation error of a single HGO in a similar way. Hence, one can compare (3) with (8) and see that single HGO's estimation is equal to MHGO estimation with the same initial condition and fixed $\hat{\alpha}_i$s; in other words $\hat{x}(t) = \sum_{i=1}^N \hat{\alpha}_i(0)\hat{x}_i(\alpha_0, t)$ where $\alpha_0 = \begin{bmatrix}\hat{\alpha}_1(0) & \cdots & \hat{\alpha}_N(0)\end{bmatrix}^T$. Thus, $\eta_s = D(\epsilon)^{-1}\tilde{x}$ is $\eta_s(t) = E_1(t)\tilde{\bar{\alpha}}_0 + \sigma(\hat{\bar{\alpha}}_0, t)$ where $\sigma(\hat{\bar{\alpha}}_0, t) = \eta_N(\hat{\bar{\alpha}}_0, t) - \eta_N(\bar{\alpha}^*, t)$. Now similar to MHGO, by choosing $0 < \epsilon < \epsilon_1^*$, the following equation can be obtained

$$\sup_{0\leq\tau\leq t}\|\eta_s(\tau)\| \leq \frac{k_1\|E_1(0)\|\|\tilde{\bar{\alpha}}_0\|}{1 - \frac{\epsilon}{\lambda_1}k_1L_1} \tag{42}$$

One can consider (41) and (42) and conclude that for the same $\epsilon$, by choosing $\gamma$ big enough, the proposed MHGO can result in a better estimation.

## III. Simulation Results

In this section, simulation is performed on a pendulum to demonstrate the capability of the proposed MHGO method in providing accurate state estimations. The equation of motion of a pendulum in the tangential direction is as follows [2]:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta} \tag{43}$$

where $m$ is the mass of the bob, $l$ is the length of the rod, $\theta$ is the angle, $g$ is the acceleration due to gravity, and $k$ is the coefficient of friction. The state variables are $x_1 = \theta$, $x_2 = \dot{\theta}$, and $y = \theta$ is the output of the system. The parameters and initial condition of system states are selected as $l = g/10$, $m = k$, and $x(0) = \begin{bmatrix}\pi/2 & 0\end{bmatrix}^T$. To estimate the system states, three HGOs with the initial conditions of $\hat{x}_1(0) = \begin{bmatrix}+5 & +5\end{bmatrix}^T$, $\hat{x}_2(0) = \begin{bmatrix}-5 & +5\end{bmatrix}^T$, and $\hat{x}_3(0) = \begin{bmatrix}+5 & -5\end{bmatrix}^T$ and design parameters $\kappa_1 = 2$, $\kappa_2 = 1$, and $\epsilon = 0.05$ are utilized. It is obvious that $x(0)$ is in the convex hull of $\hat{x}_i(0)$s. Furthermore, a saturated version of $f(x)$ is employed as $f_0(\hat{x}_o) = 200\tanh(f(\hat{x}_o)/200)$.

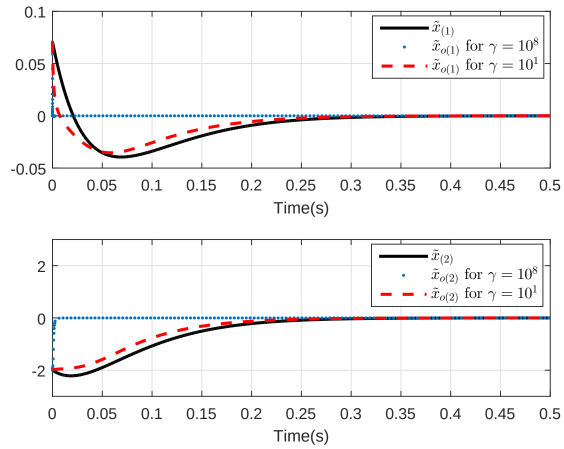


Fig. 1: Estimation error of pendulum states using one HGO and MHGO for $\epsilon = 0.05$.

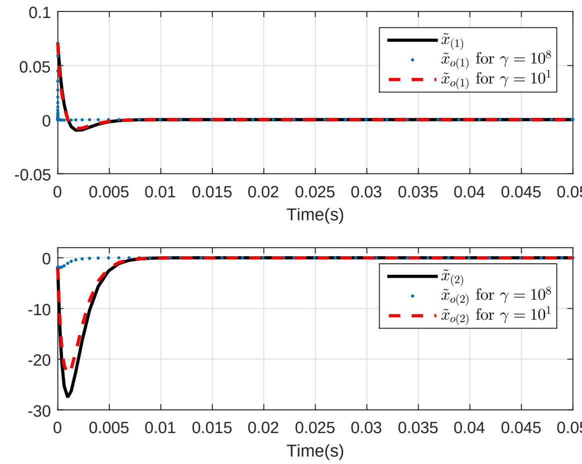


Fig. 2: Estimation error of pendulum states using one HGO and MHGO for $\epsilon = 0.001$.

To estimate $\alpha_i^*$s, the RLS algorithm with the initial conditions $\hat{\bar{\alpha}}_0 = \begin{bmatrix}0.35 & 0.35\end{bmatrix}^T$ and $P(0) = \gamma I_2$ is utilized. In order to demonstrate that by choosing $\gamma$ big, the proposed MHGO results in better estimation, the simulation is performed for two different values, i.e., $\gamma = 10^1$ and $\gamma = 10^8$. Moreover, to compare the performance of the proposed strategy to single HGO, simulation results are also provided using a single HGO with the same $f_0$, design parameters $\kappa_i$s and $\epsilon$, and the initial condition, i.e., $\hat{x}(0) = \sum_{i=1}^{3} \hat{\alpha}_i(0)\hat{x}_i(0) = \begin{bmatrix}1.5 & 2\end{bmatrix}^T$. Fig. 1 depicts the estimation error of the single HGO, $\tilde{x}(t)$, compared to the estimation errors of MHGO. It can be seen that the proposed method results in a more desirable estimation when $\gamma$ is chosen big.

To show that not only MHGO estimation converges fast, but also it has solved the peaking problem, the simulation is also performed for $\epsilon = 0.001$. The simulation results for single HGO and MHGO with $\gamma = 10^1$ and $\gamma = 10^8$ are expressed in Fig. 2. It can be seen that choosing $\epsilon$ small results in the fast convergence of HGO estimation, but it also produces undesirable peaks. Furthermore, for MHGO, choosing $\gamma$ big enough solved the peaking problem and also the estimation converged faster.

## IV. Conclusions

A new MHGO method was presented to find state estimations of nonlinear systems. The method combines estimations provided by multiple HGOs to obtain a more accurate estimation. The existence of required parameters was shown, and they were estimated using the RLS algorithm. Moreover, it was guaranteed that state estimations converge to the state of the plant, and it yields better transient response. The simulation results demonstrate that the consequence of using the proposed observer is a more accurate state estimation with more preferable transient response in comparison to single HGO.

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